General Relativistic Contribution to Polarization of the Cosmic Microwave Background Anisotropy

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Abstract

We solve the wave equation for the electromagnetic field tensors associated with the CMBR photons in a universe with scalar metric perturbations. We show that the coupling of the electromagnetic fields with the curvature associated with the scalar perturbations gives rise to an optical rotation of the microwave background photons. The magnitude of the gravitationally generated V-Stokes parameter anisotropy Δ_V, is however very small compared to the linear polarisation caused by Thomson scattering.

In the standard treatment of the microwave background anisotropies [1] it is assumed that the gravitational potential at the last scattering surface gives rise to temperature anisotropies at large angular scales through the Sachs-Wolfe effect [2]. Thomson scatterings near the surface of last scattering cause a linear polarisation of the CMB photons.

The anisotropies of CMBR are governed by the the following set of equations: The Boltzmann equation for the photon ditribution function $f(p_i, p_0, z)$

$$\frac{Df}{D\lambda} = p^0 \frac{\partial f}{\partial x^0} + p^i \frac{\partial f}{\partial x^i} + \frac{dp^i}{d\lambda}\frac{\partial f}{\partial p^i} = 0 \tag{1}$$

the geodesic equation of photons in the perturbed metric

$$\frac{dp^\alpha}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma} p^\beta p^\gamma = 0 \tag{2}$$

and the photon disperison relation

$$g_{\mu\nu} p^\mu p^\nu = 0. \tag{3}$$

The dispersion relation (3) is derived in the zeroeth order of eikonal approximation from the electromagnetic field wave equation in a curved spacetime. In the standard eikonal approximation of photon trajectories in gravitational fields, the red shift of the photon is independent of the polarization of its spin and gravity therefore makes no contribution to the polarization of the CMB radiation fields.

It has been shown in [3]that the gauge invariant wave equation of electromagnetic fields in curved spacetimes is given by

$$\nabla^\mu \nabla_\mu F_{\nu\lambda} + R_{\rho\mu\nu\lambda}\, F^{\rho\mu} + R^\rho{}_\lambda F_{\nu\rho} - R^\rho{}_\lambda F_{\lambda\rho} = 0 \tag{4}$$

Electromagnetic field tensors couple to the Riemann and Ricci curvature tensors of the background spacetime and these couplings can give rise to polarisation of electromagnetic waves in gravitational fields. In this we show that scalar perturbations to the Robertson-Walker metric near the surface of last scattering can give rise to circular polarization to the CMB radiation. We calculate the anisotropy in this gravitationally generated circular polarisation as a function of angular separation of the beams. We find however that the magnitude of this circular polarization anisotropy is unlikely to be observed in the planned CMB polarisation experiments [4].

We consider the scalar perturbations in the Friedmann- Robertson-Walker Universe described by the metric [5]

$$ds^2 = a^2(\eta)\left[(1+2\psi)d\eta^2 - (1-2\phi)\gamma_{ij}dx^i dx^j\right] \tag{5}$$

In the synchronous gauge the two scalar perturbation are the Newtonian potential ψ and the spatial curvature ϕ. The eikonal equation obeyed by the photon wave vector $q_\mu = (q_o, \vec{q})$ which follows from the wave equation (1) is given by [3]

$$\left(q^2 \cdot \delta^i{}_j + \epsilon^i{}_j\right) f_{oi} = 0 \tag{6}$$

where f_{oi} is the amplitude of the electric field associated with the radiation and the anisotropic gravitational permeability tensor $\epsilon^i{}_j$ is given by [3]

$$\epsilon^j{}_i = \left(R^o{}_o + R^\ell{}_o \ \tfrac{q_\ell}{q_o}\right)\delta^j{}_i + \left(-2R^{oj}{}_{oi} + 4R^{\ell j}{}_{oi} \ \tfrac{q_\ell}{q_o} + R^j{}_i - R^j{}_o \ \tfrac{q_i}{q_o}\right) \tag{7}$$

The components of the permeability tensor $\epsilon^j{}_i$ which follows from the metric (2) are given in Appendix A.

Consider a photon wavenumber q and choose the z-axis along $\vec{q}$. The wave equations for the transverse $\vec{E}$ field components is

$$\begin{pmatrix} q^2 + \epsilon^1{}_1 & \epsilon^2{}_1 \\ \epsilon^1{}_2 & q^2 + \epsilon^2{}_2 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = 0 \tag{8}$$

where

$$\epsilon^1{}_1 = -\frac{8\pi G}{3}\bar{\rho}(1-2\psi) + \frac{2}{a^2}\left(k^1 k_1\right)\psi\left(\vec{k},\eta\right) \tag{9}$$

$$\epsilon^2_1 = \frac{2}{a^2}k^2 k_1 \psi\left(\vec{k},\eta\right) \tag{10}$$

and similarly for the others, where $\vec{k} = (k_1, k_2, k_3)$ is the wavenumber of the gradients of the scalar perturbation $\psi(\vec{k},\eta)$. The dispersion relations of the two propagating modes $E_\pm$ are obtained by setting the determinant of the matrix operator (5) to zero to give

$$q_{0\,\pm}^2 = q^2 \left[1 - \frac{\epsilon^1{}_1 + \epsilon^2{}_2}{2q^2} \pm \frac{1}{2q^2} \left(\left(\epsilon^1{}_1 - \epsilon^2{}_2 \right)^2 + 4\epsilon^1{}_2 \epsilon^2{}_1 \right)^{1/2} \right] \tag{11}$$

The eigenvectors $E_\pm$ are given in terms of the mixing angle χ as

$$\begin{pmatrix} E_+ \\ E_- \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \tag{12}$$

where the mixing angle χ is given by

$$\tan 2\chi = \frac{2\epsilon^1{}_2 \epsilon^2{}_1}{(\epsilon^1{}_1 - \epsilon^2{}_2)} \tag{13}$$

Expressing $\epsilon^i{}_j$ in terms of polar coordinates of $\vec{k}$, we have

$$q_{0\pm} = q[1 + \frac{8\pi G}{3q^2} \bar{\rho}(1 - 2\psi) \mp \frac{k^2 sin^2\theta\ \psi}{q^2 a^2}]^{1/2} \tag{14}$$

and the mixing angle as defined in (13) is

$$\tan 2\chi = \tan 2\phi \tag{15}$$

The normal mode solutions $E_\pm$ are therefore given by

$$\begin{aligned} E_+ &= [E_1 \cos\phi + E_2 \sin\phi]\, e^{i((q_{0+}\eta) - qz)} \\ E_- &= [-E_1 \sin\phi + E_2 \cos\phi]\, e^{i((q_{0-}\eta) - qz)} \end{aligned} \tag{16}$$

The Stokes parameters of a radiation field are described by the 2×2 complex matrix [5],

$$\rho_{ij} = \frac{\left\langle E_i E_j^* \right\rangle}{I} = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix} \tag{17}$$

where

$$\begin{aligned} &I \equiv \left\langle E_+ E_+{}^* + E_- E\ {}^* \right\rangle \\ &Q \equiv \left\langle E_+ E_+{}^* - E_- E\ {}^* \right\rangle \\ &U \equiv \left\langle E_+ E_-{}^* + E_- E_+{}^* \right\rangle; \\ &V = i \left\langle E_+ E_-{}^* - E_+{}^* E_- \right\rangle \end{aligned} \tag{18}$$

where the time average is taken over duration larger than the inverse frequency. In unpolarised radiation $Q = U = V = 0$. The degree of polarisation is defined by a vector $\vec{P}$ with magnitude $P = \sqrt{Q^2 + U^2 + V^2}$. The quantity $\ell = \sqrt{Q^2 + U^2}$

denotes the degree of linear polarization and V denotes the circular polarisation of the radiation. Using the eigenmodes in the gravitational field given by (16) and using the definitions (18) we find the Stokes parameters of the propagating beam are

$$\begin{aligned} Q(\eta) &= Q\,(\eta_i)\cos 2\phi + U(\eta_i) sin 2\phi \\ U(\eta) &= U\,(\eta_i)\cos 2\phi\; cos\delta(\eta-\eta_i) - Q\,(\eta_i)\sin 2\phi\cos(\Delta\omega(\eta-\eta_i)) \\ V(\eta) &= (-U\,(\eta_i)\cos 2\phi + Q\,(\eta_i)\sin 2\phi)\sin(\Delta\omega(\eta-\eta_i)) \\ I\,(\eta) &= I\,(\eta_i) \end{aligned} \tag{19}$$

where

$$\Delta\omega \equiv (q_{0+} - q_{0-}) = \frac{k^2}{a^2 q}\sin^2\theta\psi(k,\eta) \tag{20}$$

is the rate at which the plane of polarisation is rotated.

We see that if at some initial time η_i there is non-zero $Q\,(\eta_i)$ or $U\,(\eta_i)$ then at a later time a circular polarization $V(\eta)$ is generated. The intensity of the beam $I(\eta)$ remains unchanged. One can express (16) as differential equations for evolution of Q, U, V as

$$\begin{aligned} \tfrac{\partial U}{\partial \eta} &= (\Delta\omega)\sin(\Delta\omega\eta)\,[U\cos 2\phi - Q\sin 2\phi] \\ \tfrac{\partial Q}{\partial \eta} &= 0 \\ \tfrac{\partial V}{\partial \eta} &= (\Delta\omega)\cos(\Delta\omega\eta)\,[-U\cos 2\phi + Q\sin 2\phi] \end{aligned} \tag{21}$$

To the leading order in $(\Delta\omega)$ we see that $\dot{U} = \dot{Q} = 0$ and

$$\frac{\partial V}{\partial \eta} = (\Delta\omega)\,[U\cos 2\phi - Q\sin 2\phi] \tag{22}$$

Assuming that at the time of decoupling η_i there is a non-zero $U\,(\eta_i)\;\; Q\,(\eta_i)$ due to Thomson scattering, one can estimate the degree of circular polarisiation V in CMB radiation.

From the form of equation (21) it is clear that the circular polarisation V averaged over all angles ϕ of the gravitational perturbation vector $\vec{k}$ will be zero. The rms value of V defined as $\Delta_V(q,k,cos\theta) =< (V - < V >)^2 >^{1/2}$. Similarly defining Δ_Q and, $\Delta_U)$, we see that the polarization anisotropies obey the coupled set of Boltzmann equations [6],

$$\dot{\Delta}_U + ik\mu\Delta_U = -\dot{\kappa}\Delta U \tag{23}$$

$$\dot{\Delta}_Q + ik\mu\Delta_Q = -\dot{\kappa}\left[\Delta_Q - \frac{1}{2}\left(1 - P_2(\mu)\right) S_P\right] \tag{24}$$

$$\dot{\Delta}_V + ik\mu\Delta_V = \left(\Delta_Q \sin 2\phi\right)\left(\Delta\omega\right) \tag{25}$$

where $\mu = \cos\theta$ and $S_P = -\left(\frac{5}{2}\right)\Delta T_2$ the quadrupole temperature anisotropy. The gravitational contribution arises as a source term for the V-polarisation mode while the Q and U modes are generated by Thomson scattering parametrised by $\dot{\kappa} \equiv (\chi_e n_e \sigma_T a(\eta)/a\left(\eta_o\right))$ with χ_e the ionised fraction, n_e the electron number density and σ_T the Thomson scattering cross section. In the tight coupling regime (keeping only leading order terms in $\dot{\kappa}^{-1}$) the solutions of (23) and (24) are given by

$$\Delta_U = 0 \tag{26}$$

$$\Delta_Q = -\frac{15}{8}\sin^2\theta\Delta_{T_2} \tag{27}$$

where we see that the Q polarization is generated by a quadrupole temperature anisotropy Δ_{T_2}.

Using (25) and (27) the solution of $\Delta_V\left(k,\eta,\mu\right)$ is

$$\Delta_V\left(k,\eta,\mu\right) = e^{-i\mu(\eta-\eta_i)}\left(\frac{-15}{8}\right)\sin^2\theta\Delta_{T_2}\Delta\omega\left(\eta-\eta_i\right) \tag{28}$$

One can estimate that the two point correlations

$$C^{VT}(\theta) \equiv \left\langle\Delta_T\left(\hat{n}_1\right)\Delta_V\left(\hat{n}_2\right)\right\rangle \tag{29}$$

and

$$C^{TT}(\theta) \equiv \left\langle\Delta_T\left(\hat{n}_1\right)\Delta_T\left(\hat{n}_2\right)\right\rangle \tag{30}$$

have relative magnitudes

$$\frac{C^{VT}(\theta)}{C^{TT}(\theta)} \simeq \Delta\omega\cdot\left(\eta-\eta_i\right) \simeq \frac{k_{max}^2}{q}H_o^{-1} \tag{31}$$

Taking the photons wavelength $\sim 1m$ and the smallest measurable metric perturbation to be the size of galaxies $\sim 100kpc$, we find that the GTR contribution to polarisation anisotrpy is smaller than the temperature anisotropy by a factor of 10^{-41}. This may be compared with the corresponding factor for the ratio of the polarization anisotropy due to Thomson scattering to temperature anisotropy which is 10^{-7}.

This ratio is very small unless the anisotropies are observed at very small angles (large K_{max}) but here the observations are difficult because of contamination from point sources. In conclusion, we note that gravitational couplings of the scalar perturbations on photon polarisations can generate circular polarisation anisotropies in principle. However, in practice, it would be difficult to observe.

Appendix

For the metric (2) with $\phi = \psi$ Riemann and Ricci components of curvature are

$$\begin{aligned}
R^{ij}{}_{k\ell} &= \tfrac{1}{a^2}\left[\psi^i{}_\ell \delta^j{}_k - \psi^i{}_k \delta^j{}_\ell + \psi^j{}_k \delta^i{}_\ell - \psi^j{}_\ell \delta^i{}_k\right] \\
&= -\tfrac{\dot a}{a^3}\left[\tfrac{\dot a}{a}(1-2\psi) - 2\dot\psi\right]\left(\delta^i{}_k \delta^j{}_\ell - \delta^i{}_\ell \delta^j{}_k\right) \\
R^{oi}{}_{oj} &= \tfrac{1}{a^2}\left\{\psi^i{}_j - \delta^i{}_j\left[\left(\tfrac{\ddot a}{a} - \tfrac{\dot a^2}{a^2}\right)(1-2\psi) - \tfrac{2\dot a\dot\psi}{a} - \ddot\psi\right]\right. \\
R^{ij}{}_{ok} &= \tfrac{1}{a^2}\left\{\left(\dot\psi^j + \tfrac{\dot a}{a}\psi^j\right)\delta^i{}_k - \left(\dot\psi^i + \tfrac{\dot a}{a}\psi^i\right)\delta^j{}_k\right\} \\
R^i{}_j &= -\tfrac{1}{a^2}\delta^i{}_j\left\{\left(\tfrac{\ddot a}{a} + \tfrac{\dot a^2}{a^2}\right)(1-2\psi) - \ddot\psi - \tfrac{6\dot a\dot\psi}{a} + \psi^k{}_k\right\} \\
R^o{}_o &= \tfrac{1}{a^2}\left\{\psi^k{}_k - 3\left[\left(\tfrac{\ddot a}{a} - \tfrac{\dot a^2}{a^2}\right)(1-2\psi) - \tfrac{2\dot a\dot\psi}{a} - \ddot\psi\right]\right. \\
R^i{}_o &= -\tfrac{2}{a^2}\left(\dot\psi^i + \tfrac{\dot a}{a}\psi^i\right)
\end{aligned}$$

and the field equations $R^i{}_j = 8\pi G\left(T^i{}_j - \frac{1}{2}\delta^i{}_j T\right)$ give

$$\begin{aligned}
\psi^k{}_k - 3\left[\left(\tfrac{\ddot a}{a} - \tfrac{\dot a^2}{a^2}\right)(1-2\psi) - \tfrac{2\dot a\dot\psi}{a} - \ddot\psi\right] &= 4\pi G\rho_b a^2(1+3\nu)(1+\delta\rho) \\
\dot\psi^i + \tfrac{\dot a}{a}\psi^i &= 4\pi G\left(\rho_b a^2\right)(1+\nu)V^i \\
\left(\tfrac{\ddot a}{a} + \tfrac{\dot a^2}{a^2}\right)(1-2\psi) - \ddot\psi - \tfrac{6\dot a\dot\psi}{a} + \psi^k{}_k &= 4\pi G\rho_b a^2(1-\nu)(1+\delta\rho)
\end{aligned}$$

wherein the equation of state $\rho = \frac{p}{\nu}$ is used.

The ϵ matrix is then given by

$$\begin{aligned}
\epsilon^k_k &= -\tfrac{2}{a^2}\left\{\psi^k_k + \tfrac{\ddot a}{a}(1-2\psi) - \ddot\psi - \tfrac{4\dot a\dot\psi}{a}\right\} \qquad \begin{matrix}(k=1,2,3)\\ \text{no summation}\end{matrix} \\
\epsilon^1_2 &= -\tfrac{2}{a^2}\psi^1_2 \\
\epsilon^2_1 &= -\tfrac{2}{a^2}\psi^2_1 \\
\epsilon^3_1 &= -\tfrac{2}{a^2}\psi^3_1 \\
\epsilon^3_2 &= -\tfrac{2}{a^2}\psi^3_2 \\
\epsilon^1_3 &= -\tfrac{2}{a^2}\left[\psi^1_3 - \tfrac{K_3}{K_o}\left(\dot\psi^1 + \tfrac{\dot a}{a}\psi^1\right)\right] \\
\epsilon^2_3 &= -\tfrac{2}{a^2}\left[\psi^2_3 - \tfrac{K_3}{K_o}\left(\dot\psi^2 + \tfrac{\dot a}{a}\psi^2\right)\right]
\end{aligned}$$

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